EIGENSTRUCTURE OF NONSELFADJOINT COMPLEX DISCRETE VECTOR STURM-LIOUVILLE PROBLEMS

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We present a study of complex discrete vector Sturm-Liouville problems, where coefficients of the difference equation are complex numbers and the strongly coupled boundary conditions are nonselfadjoint. Moreover, eigenstructure, orthogonality, and eigenfunctions expansion are studied. Finally, an example is given.

1. Introduction and motivation

Consider the parabolic coupled partial differential system with coupled boundary value conditions

$$u_t(x,t) - Au_{xx}(x,t) = 0, \quad 0 < x < 1,\ t > 0, \tag{1.1}$$

$$A_1u(0,t) + B_1u_x(0,t) = 0, \quad t > 0, \tag{1.2}$$

$$A_2u(1,t) + B_2u_x(1,t) = 0, \quad t > 0, \tag{1.3}$$

$$u(x,0) = F(x), \quad 0 \le x \le 1, \tag{1.4}$$

where $u = (u_1,u_2,\dots,u_m)^T$, $F(x)$ are vectors in $\mathbb{C}^m$, and $A,A_1,A_2,B_1,B_2 \in \mathbb{C}^{m\times m}$.

We divide the domain $[0,1]\times[0,\infty[$ into equal rectangles of sides $\Delta x = h$ and $\Delta t = l$, introduce coordinates of a typical mesh point $p = (kh, jl)$ and represent $u(kh, jl) = U(k, j)$. Approximating the partial derivatives appearing in (1.1) by the forward difference approximations

$$\begin{aligned} U_t(k,j) &\approx \frac{U(k,j+1)-U(k,j)}{l}, \\ U_x(k,j) &\approx \frac{U(k+1,j)-U(k,j)}{h}, \\ U_{xx}(k,j) &\approx \frac{U(k+1,j)-2U(k,j)+U(k-1,j)}{h^2}, \end{aligned} \tag{1.5}$$

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(1.1) takes the form

$$\frac{U(k,j+1)-U(k,j)}{l}=A\frac{U(k+1,j)-2U(k,j)+U(k-1,j)}{h^2}, \tag{1.6}$$

where $h=1/N$, $1\leq k\leq N-1$, $j\geq 0$. Let $r=l/h^2$ and we can write the last equation in the form

$$rA[U(k+1,j)+U(k-1,j)]+(I-2rA)U(k,j)-U(k,j+1)=0, \quad 1\leq k\leq N-1,\ j\geq 0, \tag{1.7}$$

where I is the identity matrix in $\mathbb{C}^{m\times m}$. Boundary and initial conditions (1.2)–(1.4) take the form

$$A_1U(0,j)+NB_1[U(1,j)-U(0,j)]=0, \quad j\geq 0, \tag{1.8}$$

$$A_2U(N,j)+NB_2[U(N,j)-U(N-1,j)]=0, \quad j\geq 0, \tag{1.9}$$

$$U(k,0)=F(kh), \quad 0\leq k\leq N. \tag{1.10}$$

Once we discretized problem (1.1)–(1.4), we seek solutions of the boundary problem (1.7)–(1.9) of the form (separation of variables)

$$U(k,j)=G(j)H(k), \quad G(j)\in\mathbb{C}^{m\times m},\ H(k)\in\mathbb{C}^{m}. \tag{1.11}$$

Substituting $U(k,j)$ given by (1.11) in expression (1.7), one gets

$$rAG(j)[H(k+1)+H(k-1)]+(I-2rA)G(j)H(k)-G(j+1)H(k)=0. \tag{1.12}$$

Let ρ be a real number and note that (1.12) is equivalent to

$$\begin{aligned} &rAG(j)[H(k+1)+H(k-1)]+G(j)H(k)-2rAG(j)H(k)\\ &\quad+\underbrace{\rho AG(j)H(k)-\rho AG(j)H(k)}-G(j+1)H(k)=0, \end{aligned} \tag{1.13}$$

or

$$rAG(j)\left[H(k+1)+\left(-2-\frac{\rho}{r}\right)H(k)+H(k-1)\right]+[(I+\rho A)G(j)-G(j+1)]H(k)=0. \tag{1.14}$$

Note that (1.14) is satisfied if sequences $\{G(j)\}$, $\{H(k)\}$ satisfy

$$G(j+1)-(I+\rho A)G(j)=0, \quad j\geq 0, \tag{1.15}$$

$$H(k+1)+\left(-2-\frac{\rho}{r}\right)H(k)+H(k-1)=0, \quad 1\leq k\leq N-1. \tag{1.16}$$

The solution of (1.15) is given by

$$G(j)=(I+\rho A)^j, \quad j\geq 0. \tag{1.17}$$

Now, we deal with boundary conditions (1.8)-(1.9). Using (1.11), we can transform them into

$$\begin{aligned} NB_1G(j)H(1)+[A_1-NB_1]G(j)H(0)=0, \quad j\geq 0,\\ [A_2+NB_2]G(j)H(N)-NB_2G(j)H(N-1)=0, \quad j\geq 0. \end{aligned} \tag{1.18}$$

By the Cayley-Hamilton theorem [7, page 206], if q is the degree of the minimal polynomial of $A\in\mathbb{C}^{m\times m}$, then for $j\geq q$, the powers $(I+\rho A)^j=G(j)$ can be expressed in terms of matrices $I,A,\dots,A^{q-1}$. So, the solutions of (1.16) and

$$NB_1A^jH(1)+[A_1-NB_1]A^jH(0)=0, \quad j=0,\dots,q-1, \tag{1.19}$$

$$[A_2+NB_2]A^jH(N)-NB_2A^jH(N-1)=0, \quad j=0,\dots,q-1, \tag{1.20}$$

are solutions of (1.16) and (1.18).

Note that (1.16) can be rewritten into

$$\Delta^2H(k-1)-\frac{\rho}{r}H(k)=0, \tag{1.21}$$

and (1.21), jointly with (1.19)-(1.20) is a strongly coupled discrete vector Sturm-Liouville problem, where ρ/r plays the role of an eigenvalue. In the last few years nonselfadjoint discrete Sturm-Liouville problems of the form (1.19)–(1.21) appeared in several situations when one using a discrete separation of variables method for constructing numerical solutions of strongly coupled mixed partial differential systems, as we could see in the above reasoning, and developments for other partial differential systems can be found in [3, 5, 6, 8]. In such papers, some eigenvalues and eigenfunctions are obtained using certain underlying scalar discrete Sturm-Liouville problem and assuming the existence of real eigenvalues for certain matrix related to the matrix coefficients arising in the boundary conditions. However, no information is given about other eigenvalues and eigenfunctions, and unnecessary hypotheses seem to be assumed due to the lack of an appropriate discrete vector Sturm-Liouville theory adapted to problems with nonselfadjoint boundary conditions.

Discrete scalar Sturm-Liouville problems are well studied [1]. The theory for the vector case is not so well developed, although for the selfadjoint case results are known in the literature, see [2, 4, 9], and recently, nonselfadjoint problem of type (1.16) with real coefficients and $q=1$ in boundary conditions (1.19)-(1.20) has been studied in [10].

This paper is devoted to the study of the eigenstructure, orthogonality, and eigenfunction expansions of the strongly coupled discrete vector Sturm-Liouville problem

$$H(k+1)-\alpha H(k)+\gamma H(k-1)=\lambda H(k), \quad 1\leq k\leq N-1, \tag{1.22}$$

$$F_{s1}H(1)+F_{s2}H(0)=0, \quad s=1,\dots,q, \tag{1.23}$$

$$L_{s1}H(N)+L_{s2}H(N-1)=0, \quad s=1,\dots,q, \tag{1.24}$$

where the unknown $H(k)$ is an m-dimensional vector in $\mathbb{C}^m$, F_{s1}, F_{s2}, L_{s1}, and L_{s2}, $s=1,\dots,q$, are matrices in $\mathbb{C}^{m\times m}$, not necessarily symmetric, α and $\gamma\neq 0$ are complex numbers, and λ is a complex parameter.

The paper is organized as follows. Section 2 deals with the existence and construction of the eigenpairs of problem (1.22)–(1.24). In Section 3, an inner product is introduced, which permits to construct an orthogonal basis in the eigenfunctions space and to obtain finite Fourier series expansions in terms of eigenfunctions. Section 4 includes a detailed example.

Throughout this paper, if $V \subset \mathbb{C}^m$, we denote by $\mathrm{LIN}(V)$ the linear hull of V.

2. Eigenstructure

We begin this section by recalling some definitions and introducing some convenient notation.

Definition 2.1. $\lambda \in \mathbb{C}$ is an eigenvalue of problem (1.22)–(1.24) if there exists a nonzero solution $\{H_\lambda(k)\}_{k=0}^{N} = H_\lambda$ of problem (1.22)–(1.24). The sequence H_λ is called an eigenfunction of problem (1.22)–(1.24) associated to λ. The pair (λ, H_λ) is called an eigenpair of the problem (1.22)–(1.24).

Definition 2.2. Given a sequence $\{f(k)\}_{k=0}^{N}$, where $f(k) \in \mathbb{C}^{p\times q}$, $k = 0,\dots,N$, and a vector subspace $W \subset \mathbb{C}^q$, denote by $\{f(k)\}_{k=0}^{N} W$ the set

$$\{f(k)\}_{k=0}^{N} W = \left\{\{f(k)P\}_{k=0}^{N},\ P \in W\right\}. \tag{2.1}$$

Note that if $\{P_1,\dots,P_n\}$ is a basis of W, then

$$\{f(k)\}_{k=0}^{N} W = \mathrm{LIN}\left(\{f(k)P_1\}_{k=0}^{N},\dots,\{f(k)P_n\}_{k=0}^{N}\right). \tag{2.2}$$

The associated algebraic characteristic equation of (1.22) is

$$z^2 - (\alpha+\lambda)z + \gamma = 0. \tag{2.3}$$

The discriminant of (2.3) is

$$\Delta = (\alpha+\lambda)^2 - 4\gamma, \tag{2.4}$$

and the solutions of (2.3) are

$$z = \frac{\alpha+\lambda \pm \sqrt{\Delta}}{2}. \tag{2.5}$$

We analyze the eigenstructure of problem (1.22)–(1.24) according to Δ.

2.1. $\Delta = 0$.

In this case, from (2.5),

$$z = \frac{\alpha+\lambda}{2} \tag{2.6}$$

is a double root, and from (2.4), we have that $(\alpha+\lambda)^2 - 4\gamma = 0$, and consequently the eigenvalues are

$$\lambda = \pm 2\sqrt{\gamma} - \alpha, \tag{2.7}$$

and the double root z is

$$z = \frac{\alpha+\lambda}{2} = \frac{\alpha \pm 2\sqrt{\gamma} - \alpha}{2} = \pm\sqrt{\gamma}. \tag{2.8}$$

So, the solutions take the form

$$\begin{aligned} H_1(k) &= (\sqrt{\gamma})^k Q_1 + k(\sqrt{\gamma})^k Q_2 = ((\sqrt{\gamma})^k I, k(\sqrt{\gamma})^k I)Q, \\ H_2(k) &= (-\sqrt{\gamma})^k Q_1 + k(-\sqrt{\gamma})^k Q_2 = ((-\sqrt{\gamma})^k I, k(-\sqrt{\gamma})^k I)Q, \end{aligned} \tag{2.9}$$

where $Q = (Q_1, Q_2)^T$ is an arbitrary complex vector of size $2m \times m$, that can be determined because the solutions $H(k) = z^k Q_1 + kz^k Q_2$, with $z = \pm\sqrt{\gamma}$, must satisfy (1.23)-(1.24), that is, for $s = 1, \dots, q$,

$$\begin{aligned} F_{s1}(zQ_1 + zQ_2) + F_{s2}Q_1 &= 0, \\ L_{s1}(z^N Q_1 + Nz^N Q_2) + L_{s2}(z^{N-1}Q_1 + (N-1)z^{N-1}Q_2) &= 0, \end{aligned} \tag{2.10}$$

or equivalently

$$\begin{aligned} (zF_{s1} + F_{s2})Q_1 + zF_{s1}Q_2 &= 0, \\ (zL_{s1} + L_{s2})Q_1 + (zNL_{s1} + (N-1)L_{s2})Q_2 &= 0. \end{aligned} \tag{2.11}$$

If we define the block matrix $M_D(z)$ of size $(2m)q \times 2m$ as

$$M_D(z) = \begin{pmatrix} zF_{11} + F_{12} & zF_{11} \\ \vdots & \vdots \\ zF_{q1} + F_{p2} & zF_{q1} \\ zL_{11} + L_{12} & zNL_{11} + (N-1)L_{12} \\ \vdots & \vdots \\ zL_{q1} + L_{q2} & zNL_{q1} + (N-1)L_{q2} \end{pmatrix}, \qquad Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \tag{2.12}$$

(2.11) can be written in a matrix form as

$$M_D(z)Q = 0. \tag{2.13}$$

If the linear system (2.13) has nontrivial solutions, for $z = \sqrt{\gamma}$ and/or $z = -\sqrt{\gamma}$, there exist solutions of the form (2.9), where $Q \in \operatorname{Ker}(M_D(z))$. We summarize the obtained result in the following theorem.

Theorem 2.3. *Let $M_D(z)$ be defined by (2.12).*

(i) *If* $\operatorname{Ker}(M_D(\sqrt{\gamma})) \neq \{0\}$, *then*

$$\left(2\sqrt{\gamma} - \alpha, \{((\sqrt{\gamma})^k I, k(\sqrt{\gamma})^k I)\}_{k=0}^{N} \operatorname{Ker}\left(M_D(\sqrt{\gamma})\right)\right) \tag{2.14}$$

is an eigenpair of Sturm-Liouville problem (1.22)–(1.24).

(ii) *If* $\operatorname{Ker}(M_D(-\sqrt{\gamma})) \neq \{0\}$, *then*

$$\left(-2\sqrt{\gamma}-\alpha, \{((-\sqrt{\gamma})^k I, k(-\sqrt{\gamma})^k I)\}_{k=0}^{N} \operatorname{Ker}\left(M_D(-\sqrt{\gamma})\right)\right) \tag{2.15}$$

is an eigenpair of Sturm-Liouville problem (1.22)–(1.24).

Definition 2.4. The eigenpairs described in Theorem 2.3 are called *type double* eigenpairs. The set of all eigenvalues corresponding to these eigenpairs will be denoted by σ_D and the corresponding eigenfunctions by B_D.

2.2. $\Delta \neq 0$. If $\Delta \neq 0$, from (2.5) the two different roots are

$$z_1 = \frac{\alpha+\lambda+\sqrt{\Delta}}{2}, \qquad z_2 = \frac{\alpha+\lambda-\sqrt{\Delta}}{2}, \tag{2.16}$$

and the solutions, in this case, take the form

$$H(k) = z_1^k Q_1 + z_2^k Q_2 = (z_1^k I, z_2^k I)Q, \tag{2.17}$$

where $Q = (Q_1, Q_2)^T$ is an arbitrary complex vector of size $2m \times m$. The solution $H(k)$ of (2.17) must satisfy (1.23)-(1.24), that is, for $s = 1, \dots, q$,

$$\begin{gathered} F_{s1}(z_1 Q_1 + z_2 Q_2) + F_{s2}(Q_1+Q_2) = 0, \\ L_{s1}(z_1^N Q_1 + z_2^N Q_2) + L_{s2}(z_1^{N-1}Q_1 + z_2^{N-1}Q_2) = 0, \end{gathered} \tag{2.18}$$

or equivalently

$$\begin{gathered} (z_1 F_{s1} + F_{s2})Q_1 + (z_2 F_{s1} + F_{s2})Q_2 = 0, \\ z_1^{N-1}(z_1 L_{s1} + L_{s2})Q_1 + z_2^{N-1}(z_2 L_{s1} + L_{s2})Q_2 = 0. \end{gathered} \tag{2.19}$$

Taking into account that z_1 and z_2 are functions of λ (see (2.16)), if we define the block matrix

$$M_S(\lambda) = \begin{pmatrix} z_1 F_{11} + F_{12} & z_2 F_{11} + F_{12} \\ \vdots & \vdots \\ z_1 F_{q1} + F_{q2} & z_2 F_{q1} + F_{q2} \\ z_1^{N-1}(z_1 L_{11} + L_{12}) & z_2^{N-1}(z_2 L_{11} + L_{12}) \\ \vdots & \vdots \\ z_1^{N-1}(z_1 L_{q1} + L_{q2}) & z_2^{N-1}(z_2 L_{q1} + L_{q2}) \end{pmatrix}, \qquad Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \tag{2.20}$$

(2.19) can be written in a matrix form as

$$M_S(\lambda)Q = 0. \tag{2.21}$$

In order to find nonzero values of Q, the linear system (2.21) has nontrivial solutions for those values of λ such that

$$\operatorname{Ker}(M_S(\lambda)) \neq \{0\}, \tag{2.22}$$

and for these values, if $Q \in \mathrm{Ker}(M_S(\lambda))$, there exist solutions $H(k)$ of the form given by (2.17).

Remark 2.5. Let $\lambda = 2\sqrt{\gamma} - \alpha$, $z = \sqrt{\gamma}$ or $\lambda = -2\sqrt{\gamma} - \alpha$, $z = -\sqrt{\gamma}$. It is possible that the type double eigenvalue λ obtained from its corresponding double root z could satisfy (2.22), and therefore, one may suppose that λ could have associated eigenfunctions different (linearly independent) from those provided by Theorem 2.3. But this fact is not true. If λ satisfies (2.22), then $z_1 = z_2 = z$ (see (2.16)), and the two block columns of $M_S(\lambda)$ are identical. So, if

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \in \mathrm{Ker}\left(M_S(\lambda)\right), \tag{2.23}$$

we obtain that

$$Q_1, Q_2 \in \mathrm{Ker}\begin{pmatrix} zF_{11} + F_{12} \\ \vdots \\ zF_{q1} + F_{q2} \\ z^{N-1}\left(zL_{11} + L_{12}\right) \\ \vdots \\ z^{N-1}\left(zL_{q1} + L_{q2}\right) \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} zF_{11} + F_{12} \\ \vdots \\ zF_{q1} + F_{q2} \\ zL_{11} + L_{12} \\ \vdots \\ zL_{q1} + L_{q2} \end{pmatrix}. \tag{2.24}$$

Consequently, $(Q_1, 0), (Q_2, 0) \in \mathrm{Ker}(M_D(z))$ and the eigenfunctions obtained from expression (2.17) are

$$H(k) = z^k Q_1 + z^k Q_2 = z^k\left(Q_1 + Q_2\right) = z^k Q, \quad Q \in \mathrm{Ker}\left(M_D(z)\right), \tag{2.25}$$

included in the set of those given by Theorem 2.3. So, type double eigenvalues have to be removed from the values of λ that satisfy (2.22) because their corresponding eigenfunctions are only some of the set of type double eigenfunctions.

Theorem 2.6. *Let $M_S(\lambda)$ be defined by (2.20), and let $\{\lambda_1, \ldots, \lambda_r\}$ be complex values satisfying*

$$\mathrm{Ker}\left(M_S(\lambda_i)\right) \neq \{0\}, \tag{2.26}$$

with the exception of $\pm 2\sqrt{\gamma} - \alpha$. So,

$$\left(\lambda_i, \left\{\left(z_1(\lambda_i)^k I, z_2(\lambda_i)^k I\right)\right\}_{k=0}^{N} \mathrm{Ker}\left(M_S(\lambda_i)\right)\right), \tag{2.27}$$

for $i = 1, \ldots, r$, are eigenpairs of Sturm-Liouville problem (1.22)–(1.24), where

$$\begin{aligned} z_1(\lambda_i) &= \frac{\alpha + \lambda_i + \sqrt{(\alpha + \lambda_i)^2 - 4\gamma}}{2}, \\ z_2(\lambda_i) &= \frac{\alpha + \lambda_i - \sqrt{(\alpha + \lambda_i)^2 - 4\gamma}}{2}. \end{aligned} \tag{2.28}$$

Theorem 2.6 suggests the introduction of the following concept.

Definition 2.7. With the notation of Theorem 2.6, the possible eigenpairs described in (2.27) will be called *type simple* eigenpairs. The set of all eigenfunctions corresponding to the type simple eigenpairs will be denoted by B_S and the eigenvalues by elements of σ_S.

Summarizing, all the conclusions of this section are contained in the following result.

THEOREM 2.8. *Consider the hypotheses and notation of Theorems 2.3 and 2.6. Let* $\sigma = \sigma_D \cup \sigma_S$ *and* $B = B_D \cup B_S$.

(1) *The Sturm-Liouville problem (1.22)–(1.24) admits nontrivial solutions if and only if* $\sigma \neq \emptyset$.
(2) *If* $\sigma \neq \emptyset$, *every eigenfunction of problem (1.22)–(1.24) is a linear combination of the eigenfunctions of* B.

Remark 2.9. In practice, it is more usual to work with real coefficients. This fact leads to the following result. Consider Sturm-Liouville problem (1.22)–(1.24), suppose that $\alpha, \gamma \in \mathbb{R}$, $F_{s1}, F_{s2}, L_{s1}, L_{s2} \in \mathbb{R}^{m\times m}$ for $s = 1, \dots, q$, and let

$$\left(\lambda, \{f(k) + ig(k)\}_{k=0}^{N}\right) \tag{2.29}$$

be an eigenpair of (1.22)–(1.24), $f(k), g(k) \in \mathbb{R}$, $0 \le k \le N$. If $\lambda \in \mathbb{R}$, it is easy to show that

$$\left(\lambda, \{f(k)\}_{k=0}^{N}\right), \qquad \left(\lambda, \{g(k)\}_{k=0}^{N}\right) \tag{2.30}$$

are eigenpairs of (1.22)–(1.24).

3. Orthogonality and eigenfunction expansions

Consider the notation of Section 2 and denote by SL the vector space of the solutions of Sturm-Liouville problem (1.22)–(1.24) that by Theorem 2.8 is the set of all linear combinations of eigenfunctions of B. The aim of this section is to obtain an explicit representation of a given function $\{f(k)\}_{k=0}^{N}$ in SL in terms of eigenfunctions of B. This task implies solving a linear system. But having some orthogonal structure in B, we would determine the coefficients of the linear expansion as Fourier coefficients, which are much more interesting from a computational point of view. A possible orthogonal structure of SL is available using Gram-Schmidt orthogonalization method to the set of eigenfunctions B given in Theorem 2.8, endowing to B of an inner product structure, which recover the properties of scalar discrete Sturm-Liouville problems, see [1, pages 664–666].

Consider the usual inner product in $\mathbb{C}^m$, that is, $\langle\cdot,\cdot\rangle : \mathbb{C}^m \times \mathbb{C}^m \longrightarrow \mathbb{C}$ such that $\langle u, v\rangle = u^T\overline{v}$ for all $u, v \in \mathbb{C}^m$ and we define an inner product in SL as follows: if $\phi_\mu = \{\phi_\mu(k)\}_{k=0}^{N}$, $\phi_\lambda = \{\phi_\lambda(k)\}_{k=0}^{N}$ are in SL,

$$[\phi_\mu, \phi_\lambda] = \sum_{k=1}^{N-1} \langle \phi_\mu(k), \phi_\lambda(k)\rangle. \tag{3.1}$$

The eigenfunctions obtained in Section 2 are linear combinations of discrete functions of the form $\{f(k)P\}_{k=0}^{N}$, where $f(k) \in \mathbb{C}$ for $0 \le k \le N$, and $P \in \mathbb{C}^m$. This fact motivates the following result.

Corollary 3.1. *If P, Q are orthogonal vectors in $\mathbb{C}^m$ and $f(k)$, $g(k)$ are complex numbers for $0 \le k \le N$, then $[\{f(k)P\}_{k=0}^{N}, \{g(k)Q\}_{k=0}^{N}] = 0$.*

Proof. By definition (3.1),

$$\left[\{f(k)P\}_{k=0}^{N}, \{g(k)Q\}_{k=0}^{N}\right] = \sum_{k=1}^{N-1} \langle f(k)P, g(k)Q\rangle = \sum_{k=1}^{N-1} f(k)\overline{g(k)}\langle P, Q\rangle = 0. \tag{3.2}$$

□

As we indicated before, using the inner product (3.1), we can orthogonalize the eigenfunctions of B by means of the Gram-Schmidt orthogonalization method. So, we can state, without proof, the vector analogue of the Fourier series expansion in terms of an orthogonal basis of SL, see [1, page 675].

Corollary 3.2. *Let $T = \{\tau_1, \dots, \tau_n\}$ be an orthogonal basis of* SL *with respect to the inner product (3.1). Let $f = \{f(k)\}_{k=0}^{N} \in$ SL, then*

$$f(k) = \sum_{s=1}^{n} \alpha_s \tau_s(k), \quad \alpha_s = \frac{[\tau_s, f]}{[\tau_s, \tau_s]}, \ 1 \le s \le n, \tag{3.3}$$

and coefficients $\alpha_s \in \mathbb{C}$, are called the Fourier coefficients of f with respect to T.

4. Example

We consider the parabolic coupled partial differential system (1.1)–(1.4), where

$$\begin{gathered} A = \begin{pmatrix} -5 & -3 \\ -10 & -9 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} -10 & 7 \\ -9 & 2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 & -5 \\ -1 & 7 \end{pmatrix}, \\ B_1 = \begin{pmatrix} 5 & 3 \\ -5 & 10 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 3 & -6 \\ 2 & 8 \end{pmatrix}. \end{gathered} \tag{4.1}$$

For $N = 5$ and taking into account that the degree of minimal polynomial of A is $q = 2$, the discretization and separation of variables method of Section 1 lead to the discrete Sturm-Liouville problem

$$\begin{gathered} H(k+1) + \left(-2 - \frac{\rho}{r}\right)H(k) + H(k-1) = 0, \quad 1 \le k \le 4, \\ 5B_1H(1) + [A_1 - 5B_1]H(0) = 0, \\ 5B_1AH(1) + [A_1 - 5B_1]AH(0) = 0, \\ [A_2 + 5B_2]H(5) - 5B_2H(4) = 0, \\ [A_2 + 5B_2]AH(5) - 5B_2AH(4) = 0. \end{gathered} \tag{4.2}$$

This problem is a vector discrete Sturm-Liouville problem of the type (1.22)–(1.24), where $N = 5$, $\alpha = 2$, $\gamma = 1$, $\lambda = \rho/r$, and

$$\begin{gathered} F_{11} = 5B_1 = \begin{pmatrix} 25 & 15 \\ -25 & 50 \end{pmatrix}, \qquad F_{12} = A_1 - 5B_1 = \begin{pmatrix} -35 & -8 \\ 16 & -48 \end{pmatrix}, \\ F_{21} = 5B_1A = \begin{pmatrix} -275 & -210 \\ -375 & -375 \end{pmatrix}, \qquad F_{22} = [A_1 - 5B_1]A = \begin{pmatrix} 255 & 177 \\ 400 & 384 \end{pmatrix}, \\ L_{11} = A_2 + 5B_2 = \begin{pmatrix} 17 & -35 \\ 9 & 47 \end{pmatrix}, \qquad L_{12} = -5B_2 = \begin{pmatrix} -15 & 30 \\ -10 & -40 \end{pmatrix}, \\ L_{21} = [A_2 + 5B_2]A = \begin{pmatrix} 265 & 264 \\ -515 & -450 \end{pmatrix}, \qquad L_{22} = -5B_2A = \begin{pmatrix} -225 & -225 \\ 450 & 390 \end{pmatrix}. \end{gathered} \tag{4.3}$$

First, we try to find the type double eigenfunctions. So,

$$M_D(z) = \begin{pmatrix} -275+25z & -210+15z & 25z & 15z \\ -375-25z & -375+50z & -25z & 50z \\ 255-35z & 177-8z & -35z & -8z \\ 400+16z & 384-48z & 16z & -48z \\ 265+17z & 264-35z & 1060+85z & 1056-175z \\ -515+9z & -450+47z & -2060+45z & -1800+235z \\ -225-15z & -225+30z & -900-75z & -900+150z \\ 450-10z & 390-40z & 1800-50z & 1560-200z \end{pmatrix}, \tag{4.4}$$

and for $z = \pm\sqrt{\gamma} = \pm 1$, we have that $\operatorname{Ker}(M_D(z)) = \{0\}$. Therefore, from Theorem 2.3, there are no eigenvalues and no eigenfunctions of type double.

For type simple eigenfunctions, we first compute the blockmatrix $M_S(\lambda)$, and following Theorem 2.6 the complex values such that $\operatorname{Ker}(M_S(\lambda)) \neq \{0\}$, except $\pm 2\sqrt{\gamma} - \alpha = \pm 2 \times 1 - 2 = \{-4, 0\}$, are

$$\{-2, -2-\sqrt{2}, -2+\sqrt{2}\}. \tag{4.5}$$

So,

(1) for $\lambda_1 = -2$, we have

$$\begin{gathered} z_1(\lambda_1) = i, \qquad z_2(\lambda_1) = -i, \\ \operatorname{Ker}(M_S(\lambda_1)) = \left\langle \begin{matrix} (-3+3i, -10-6i, 0, 14), \\ (-3-5i, -5+5i, 7, 0) \end{matrix} \right\rangle, \end{gathered} \tag{4.6}$$

and the associated eigenfunctions are given by

$$\begin{gathered} \tau^1_{\lambda_1}(k) = i^k \begin{pmatrix} -3+3i \\ -10-6i \end{pmatrix} + (-i)^k \begin{pmatrix} 0 \\ 14 \end{pmatrix}, \\ \tau^2_{\lambda_1}(k) = i^k \begin{pmatrix} -3-5i \\ -5+5i \end{pmatrix} + (-i)^k \begin{pmatrix} 7 \\ 0 \end{pmatrix}; \end{gathered} \tag{4.7}$$

(2) for $\lambda_2 = -2 - \sqrt{2}$, we have

$$z_1(\lambda_2) = \frac{-1+i}{2}, \qquad z_2(\lambda_2) = \frac{-1-i}{2},$$
$$\operatorname{Ker}(M_S(\lambda_2)) = \left\langle \left(\frac{3((1-i)+45\sqrt{2})}{(59+675i)+(323+322i)\sqrt{2}}, -\frac{(16+16i)+(9+5i)\sqrt{2}}{2((8+7i)+7\sqrt{2})}, 0, 1 \right), \left(\frac{(-5-729i)+(82-323i)\sqrt{2}}{(59+675i)+(323+322i)\sqrt{2}}, \frac{(5-5i)\sqrt{2}}{(8+7i)+7\sqrt{2}}, 1, 0 \right) \right\rangle, \tag{4.8}$$

and the associated eigenfunctions are given by

$$\tau^1_{\lambda_2}(k) = \left(\frac{-1+i}{2}\right)^k \begin{pmatrix} \dfrac{3((1-i)+45\sqrt{2})}{(59+675i)+(323+322i)\sqrt{2}} \\ -\dfrac{(16+16i)+(9+5i)\sqrt{2}}{2((8+7i)+7\sqrt{2})} \end{pmatrix} + \left(\frac{-1-i}{2}\right)^k \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$\tau^2_{\lambda_2}(k) = \left(\frac{-1+i}{2}\right)^k \begin{pmatrix} \dfrac{(-5-729i)+(82-323i)\sqrt{2}}{(59+675i)+(323+322i)\sqrt{2}} \\ \dfrac{(5-5i)\sqrt{2}}{(8+7i)+7\sqrt{2}} \end{pmatrix} + \left(\frac{-1-i}{2}\right)^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \tag{4.9}$$

(3) for $\lambda_3 = -2 + \sqrt{2}$, we have

$$z_1(\lambda_3) = \frac{1+i}{2}, \qquad z_2(\lambda_3) = \frac{1-i}{2},$$
$$\operatorname{Ker}(M_S(\lambda_3)) = \left\langle \left(\frac{(-3-3i)+135\sqrt{2}}{(59-675i)-(323-322i)\sqrt{2}}, \frac{(16-16i)-(9-5i)\sqrt{2}}{2((-8+7i)+7\sqrt{2})}, 0, 1 \right), \left(\frac{(729+5i)-(323-82i)\sqrt{2}}{(-675-59i)+(322+323i)\sqrt{2}}, \frac{(5+5i)\sqrt{2}}{(8-7i)-7\sqrt{2}}, 1, 0 \right) \right\rangle, \tag{4.10}$$

and the associated eigenfunctions are given by

$$\tau^1_{\lambda_3}(k) = \left(\frac{1+i}{2}\right)^k \begin{pmatrix} \dfrac{(-3-3i)+135\sqrt{2}}{(59-675i)-(323-322i)\sqrt{2}} \\ \dfrac{(16-16i)-(9-5i)\sqrt{2}}{2((-8+7i)+7\sqrt{2})} \end{pmatrix} + \left(\frac{1-i}{2}\right)^k \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$\tau^2_{\lambda_3}(k) = \left(\frac{1+i}{2}\right)^k \begin{pmatrix} \dfrac{(729+5i)-(323-82i)\sqrt{2}}{(-675-59i)+(322+323i)\sqrt{2}} \\ \dfrac{(5+5i)\sqrt{2}}{(8-7i)-7\sqrt{2}} \end{pmatrix} + \left(\frac{1-i}{2}\right)^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.11}$$

This finishes the search of eigenfunctions. But, note that $\alpha = 2$, $\gamma = 1$, all eigenvalues are real numbers and all matrices have only real entries. So, we can apply Remark 2.9 in order to transform the obtained eigenfunctions into real ones.

Therefore, as

$$i^k = \cos\left(\frac{k\pi}{2}\right) + i\sin\left(\frac{k\pi}{2}\right), \tag{4.12}$$

$\tau^1_{\lambda_1}(k)$ and $\tau^2_{\lambda_1}(k)$ can be transformed into

$$\begin{aligned}
\tau^1_{\lambda_1}(k) &= \cos\left(\frac{k\pi}{2}\right)\begin{pmatrix}-3\\4\end{pmatrix} + \sin\left(\frac{k\pi}{2}\right)\begin{pmatrix}-3\\6\end{pmatrix} \\
&\quad + i\left[\cos\left(\frac{k\pi}{2}\right)\begin{pmatrix}3\\-6\end{pmatrix} + \sin\left(\frac{k\pi}{2}\right)\begin{pmatrix}-3\\-24\end{pmatrix}\right], \\
\tau^2_{\lambda_1}(k) &= \cos\left(\frac{k\pi}{2}\right)\begin{pmatrix}4\\-5\end{pmatrix} + \sin\left(\frac{k\pi}{2}\right)\begin{pmatrix}5\\-5\end{pmatrix} \\
&\quad + i\left[\cos\left(\frac{k\pi}{2}\right)\begin{pmatrix}-5\\5\end{pmatrix} + \sin\left(\frac{k\pi}{2}\right)\begin{pmatrix}-10\\-5\end{pmatrix}\right],
\end{aligned} \tag{4.13}$$

and following Remark 2.9,

$$\begin{gathered}
\left(-2, \cos\left(\frac{k\pi}{2}\right)\begin{pmatrix}-3\\4\end{pmatrix} + \sin\left(\frac{k\pi}{2}\right)\begin{pmatrix}-3\\6\end{pmatrix}\right), \\
\left(-2, \cos\left(\frac{k\pi}{2}\right)\begin{pmatrix}3\\-6\end{pmatrix} + \sin\left(\frac{k\pi}{2}\right)\begin{pmatrix}-3\\-24\end{pmatrix}\right), \\
\left(-2, \cos\left(\frac{k\pi}{2}\right)\begin{pmatrix}4\\-5\end{pmatrix} + \sin\left(\frac{k\pi}{2}\right)\begin{pmatrix}5\\-5\end{pmatrix}\right), \\
\left(-2, \cos\left(\frac{k\pi}{2}\right)\begin{pmatrix}-5\\5\end{pmatrix} + \sin\left(\frac{k\pi}{2}\right)\begin{pmatrix}-10\\-5\end{pmatrix}\right)
\end{gathered} \tag{4.14}$$

are eigenpairs. In an analogous way, we can obtain the other eigenpairs. For $\lambda_2 = -2 - \sqrt{2}$,

$$\begin{aligned}
\Bigg(-2-\sqrt{2}, \frac{1}{\sqrt{2}}\Bigg[&\cos\left(k\frac{3\pi}{4}\right)\begin{pmatrix}\dfrac{3(-2730+1357\sqrt{2})}{38866}\\ \dfrac{10696-35217\sqrt{2}}{38866}\end{pmatrix} \\
&+ \sin\left(k\frac{3\pi}{4}\right)\begin{pmatrix}\dfrac{3(-406+1597\sqrt{2})}{38866}\\ \dfrac{-1790+9147\sqrt{2}}{38866}\end{pmatrix}\Bigg]\Bigg),
\end{aligned}$$

$$\left(-2-\sqrt{2},\frac{1}{\sqrt{2}}\left[\cos\left(k\frac{3\pi}{4}\right)\begin{pmatrix}\dfrac{3(-406+1597\sqrt{2})}{38866}\\ \dfrac{1790-9147\sqrt{2}}{38866}\end{pmatrix}\right.\right.$$
$$\left.\left.+\sin\left(k\frac{3\pi}{4}\right)\begin{pmatrix}\dfrac{3(-2730+1357\sqrt{2})}{38866}\\ \dfrac{-67036-5217\sqrt{2}}{38866}\end{pmatrix}\right]\right),$$
$$\left(-2-\sqrt{2},\frac{1}{\sqrt{2}}\left[\cos\left(k\frac{3\pi}{4}\right)\begin{pmatrix}\dfrac{21616-10645\sqrt{2}}{38866}\\ \dfrac{5(-2730+1357\sqrt{2})}{19433}\end{pmatrix}\right.\right.$$
$$\left.\left.+\sin\left(k\frac{3\pi}{4}\right)\begin{pmatrix}\dfrac{-3414+15535\sqrt{2}}{38866}\\ \dfrac{5(-406+1597\sqrt{2})}{19433}\end{pmatrix}\right]\right),$$
$$\left(-2-\sqrt{2},\frac{1}{\sqrt{2}}\left[\cos\left(k\frac{3\pi}{4}\right)\begin{pmatrix}\dfrac{3414-15535\sqrt{2}}{38866}\\ \dfrac{5(-406+1597\sqrt{2})}{19433}\end{pmatrix}\right.\right.$$
$$\left.\left.+\sin\left(k\frac{3\pi}{4}\right)\begin{pmatrix}\dfrac{-56116-10645\sqrt{2}}{38866}\\ \dfrac{5(-2730+1357\sqrt{2})}{19433}\end{pmatrix}\right]\right),\tag{4.15}$$

and for $\lambda_3=-2+\sqrt{2}$,

$$\left(-2+\sqrt{2},\frac{1}{\sqrt{2}}\left[\cos\left(k\frac{\pi}{4}\right)\begin{pmatrix}-\dfrac{3(2730+1357\sqrt{2})}{38866}\\ \dfrac{10696+5217\sqrt{2}}{38866}\end{pmatrix}\right.\right.$$
$$\left.\left.+\sin\left(k\frac{\pi}{4}\right)\begin{pmatrix}-\dfrac{3(406+1597\sqrt{2})}{38866}\\ \dfrac{1790+9147\sqrt{2}}{38866}\end{pmatrix}\right]\right),$$
$$\left(-2+\sqrt{2},\frac{1}{\sqrt{2}}\left[\cos\left(k\frac{\pi}{4}\right)\begin{pmatrix}\dfrac{3(406+1597\sqrt{2})}{38866}\\ \dfrac{-1790-9147\sqrt{2}}{38866}\end{pmatrix}\right.\right.$$
$$\left.\left.+\sin\left(k\frac{\pi}{4}\right)\begin{pmatrix}-\dfrac{3(2730+1357\sqrt{2})}{38866}\\ \dfrac{-67036+5217\sqrt{2}}{38866}\end{pmatrix}\right]\right),$$

$$\begin{aligned}
&\left(-2+\sqrt{2}, \frac{1}{\sqrt{2}}\left[\cos\left(k\frac{\pi}{4}\right)\begin{pmatrix} \dfrac{21616+10645\sqrt{2}}{38866} \\ -\dfrac{5(2730+1357\sqrt{2})}{19433} \end{pmatrix}\right.\right. \\
&\qquad\qquad \left.\left. +\sin\left(k\frac{\pi}{4}\right)\begin{pmatrix} \dfrac{3414+15535\sqrt{2}}{38866} \\ -\dfrac{5(406+1597\sqrt{2})}{19433} \end{pmatrix}\right]\right), \\
&\left(-2+\sqrt{2}, \frac{1}{\sqrt{2}}\left[\cos\left(k\frac{\pi}{4}\right)\begin{pmatrix} \dfrac{-3414-15535\sqrt{2}}{38866} \\ \dfrac{5(406+1597\sqrt{2})}{19433} \end{pmatrix}\right.\right. \\
&\qquad\qquad \left.\left. +\sin\left(k\frac{\pi}{4}\right)\begin{pmatrix} \dfrac{-56116+10645\sqrt{2}}{38866} \\ -\dfrac{5(2730+1357\sqrt{2})}{19433} \end{pmatrix}\right]\right).
\end{aligned} \tag{4.16}$$

The above computations were carried out using *Mathematica* [11]. Notebooks with the commented code and computations of this example, including the orthogonalization of eigenfunctions, can be obtained from http://adesur.mat.upv.es/w3/complexSL/.

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